Interaction of instantons in a gauge theory forcing their identical orientation

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Abstract

A gauge theory model in which there exists a specific interaction between instantons is considered. An effective action describing this interaction possesses a minimum when the instantons have identical orientation. The considered interaction might provide a phase transition into the state where instantons have a preferred orientation. This phase of the gauge-field theory is important because it can give the description of gravity in the framework of the gauge theory.

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The aim of this paper is to study a gauge theory model with an interaction between instantons making their identical orientation preferable. The interest to this problem is inspired by Ref. [1] where a possibility was found to describe effects of gravity in the framework of a conventional $SO(4)$ gauge theory of Yang-Mills. In this approach space-time is supposed to be basically flat. On the basic level of the theory there are only the usual for gauge theory fields: gauge bosons as well as scalars and fermions interacting with the gauge field. In order to exhibit effects of gravity the specific phenomenon, the "condensation of polarized instantons and antiinstantons", must take place in the vacuum of the $SO(4)$ gauge theory. The instantons in one $su(2)$ subalgebra of $so(4)$ gauge algebra must have the preferred direction of orientation. The antiinstantons in the other $su(2) \in so(4)$ subalgebra must also have their own proffered direction of orientation. These orientations of condensates of instantons and antiinstantons play a role of the order parameter of the considered nontrivial phase of the vacuum state. The orientations of instantons remain invariant under the local gauge transformations, see Ref. [2,3]. Therefore the possible existence of the considered nontrivial phase of the vacuum does not contradict the gauge invariance, which forbids any nontrivial state whose order parameter is noninvariant under *local* gauge transformations [4].

The most "natural" way to look for the phase with polarised instantons is to find an interaction between the instantons which forces any two instantons to have the identical orientation. Then one can expect the system to undergo a phase transition into the state with polarized instantons. Moreover, if one choose this scenario, then the antiinstantons of the same $SU(2)$ group should remain noninteracting, thus resulting in the absence of condensate of antiinstantons. The problem is that in a pure gauge theory instantons do not interact. There is the well-known interaction between instantons and antiinstantons [5] which depends on their orientation, but there are exact multi-instantons solutions with arbitrary orientations of instantons [6]. The main result of this paper is the $SU(2)$ gauge theory model providing the necessary interaction between a pair of instantons making their identical orientation most probable and giving no interaction between antiinstantons.

The qualitative description of the model is the following. Consider $SU(2)$ gauge theory with scalars and fermions. Suppose that scalars develop the condensate. Suppose also that there is

an interaction between scalars and fermions which contains scalar-pseudoscalar vertex $1-\gamma_5$. Then the right fermions are influenced by the scalar condensate while the left fermions do not interact with it. Consider now the gauge field created by several instantons. The instantons are known to interact very strongly with the fermions. Therefore the fermions give a radiative correction to the effective action which describes the instantons. Instanton influence on the fermion field most strongly manifests itself in the fermionic zero-modes [7]. In the pure gauge theory with n instantons there are n degenerate right-hand zero-modes, as follows from the index theorem [8]. The existence of the scalar condensate changes the situation drastically. For this case the fermions are influenced by the vacuum scalar field which obviously violates the condition of the index theorem (the condition is that only the pure gauge field influences upon the fermions). Therefore the zero-modes can get splitting in the field of several instantons. This splitting gives a contribution of fermions into the effective action describing the instanton field. The latter will be shown to depend on the orientation of instantons. There is no such effect for antiinstantons because their zero modes are left-handed and therefore they do not interact with the scalar condensate, satisfy the condition of the index theorem and thus exhibit no splitting. The considered interaction arises as the radiative one-fermion-loop correction to the gauge field action. Usually the radiative corrections give only the renormalization of physical quantities. For the case considered the correction results in the new kind of interaction.

In order to formulate the model we are to chose the scalar condensate in such a way that it creates the field applied to fermions V which satisfies the condition $[\gamma\nabla, V] \neq 0$, where ∇ is the covariant derivative $\gamma\nabla = \gamma_\mu(\partial_\mu - iA^a_\mu(x)T^a)$, and $T^a = \tau^a/2,\ a = 1,2,3$ are the generators of the gauge group. Otherwise no splitting of the zero modes would arise. If one wishes to consider homogeneous, x-independent field V, as is usual, then the only way to satisfy this condition is to suppose that the field V depends on the generators of the gauge group: $V \sim \vec{T}\vec{U}(1-\gamma_5)$. As a result we are to introduce into the problem some additional vector $\vec{U}$. The constant vector not only looks ugly but makes no good, as can be verified. We are to have the vector whose averaged value is zero, but the averaged values of its powers could play a role: $< U^a >= 0,\ \ < U^a U^b >= \vec{U}^2\delta_{ab},$ .... The way to do it can only be provided by the additional symmetry: we are to consider some "additional" $SU(2)$ group whose generators are

equal to the necessary vector $\vec{U}$. In this paper we will consider this $SU(2)$ group as a global one, it may be thought of as " a flavour group" or "a group of generations".

Note that there is a clear and interesting analogy between the discussed construction and the phenomenon of ferromagnetism. The instantons might be compared with the atoms with nonzero spin, the fermionic zero modes resemble the atomic outer electrons, and the scalar field which splits the zero modes plays a role very similar to the crystal field which creates the conducting band. The problem of instanton interaction, considered in the paper, looks similar to the problem of the origin of exchange integral in ferromagnetic theory.

Consider the $SU(2)$ gauge theory. Suppose that there are two generations of fermionic fields with equal masses in the fundamental representation of this gauge group. We will treat them as a doublet in the space of generations. Suppose also that there are three generations of scalars, - considered as a triplet in the space of generations, - in the vector representation of $SU(2)$ gauge group.

Let us introduce an interaction between the scalars and the right-hand fermions described by the Lagrangian

$$\mathcal{L}_{sf}(x) = f\psi^{+}_{A}(x)\Phi_i(x)U^i_{AB}[(1-\gamma_5)/2]\psi_B(x) \ , \tag{1}$$

where f is the dimensionless constant of scalar-fermion interaction, $\psi_A(x)$ is the fermion doublet, indexes $A, B = 1, 2$ label the doublet variables in the space of generations, and $\Phi_i(x), i = 1, 2, 3$ is the triplet of scalar fields. There is a freedom of choosing the matrixes $U^i = U^i_{AB}, \ i = 1, 2, 3$ describing the coupling between different generations of fermions and scalars. We choose these matrixes be the triplet of generators of rotations in the space of generations

$$U^i = U^i_{AB} = \sigma^i_{AB}/2, \ i = 1, 2, 3 \ . \tag{2}$$

The scalar fields $\Phi_i(x)$ are in the vector representation of the gauge $SU(2)$ group, therefore $\Phi_i(x) = \Phi_{i,a}(x)T^a$. Suppose now that their nonlinear self-interaction results in the development of the scalar condensate which has the following form

$$(\Phi_{i,a}(x))_{cond} = \phi\delta_{ia} \ , \tag{3}$$

where ϕ is a constant. We see that the construction presented in Eqs.(1),(2),(3) results in the desired form for the vacuum scalar field V

$$V = f\phi\ (\vec{T}\vec{U})(1-\gamma_5)/2\ , \tag{4}$$

which influences on the right-hand fermions. Note, that the Euclidean formulation is used.

Our goal is to calculate the fermionic determinant $\det(-i\gamma\nabla - im - iV)$, when the gauge field $A^a_\mu(x)$ is created by $n = 2$ instantons. It is important that the determinant depends also on the field V (4) created by the scalar condensate. Let us present it as

$$\det(-i\gamma\nabla - im - iV) = \det(-i\gamma\nabla - im)\det(F)\ ,$$

where the first factor $\det(-i\gamma\nabla - im)$ is the determinant in a pure gauge field, and is not interesting for our purposes, see discussion at the end of the paper. Only the second factor is important

$$\det(F) = \det(1 + GV) = \exp(-S_F)\ . \tag{5}$$

Here G is the propagator of the fermions in the gauge field, $G = (\gamma\nabla + m)^{-1}$, and S_F is the fermion correction to the gauge field action. The instantons create fermionic zero-modes playing a crucial role in the problem. Therefore, it is useful to distinguish them in the fermion propagator. With this purpose let us introduce the projection operator onto the states of zero modes P. It satisfies the conditions $P^2 = P$, $(\gamma\nabla)P = 0$, $\mathrm{Sp}(P) = 2n = 4$. Here $2n$ is the number of zero-modes. Remember that we consider $n = 2$ instantons and 2 generations of fermions. The propagator may be presented as $G = G_0 + G_1$ where $G_0 = P/m$, $G_1 = (1-P)(\gamma\nabla+m)^{-1}(1-P)$. To simplify calculations let us consider the case of small instantons, so that the condition $m\rho \ll 1$, where ρ is an instanton radius, is fulfilled. Then the fermionic mass m may be considered as a small parameter and we will put $m = 0$ wherever it is possible. In the limit $m = 0$ the propagator of the nonzero-modes was evaluated explicitly in Ref. [2]. Using this propagator it is easy to verify that $VG_1^{(m=0)}V = 0$. From this condition we find that nonzero modes are eliminated, only zero modes contribute to (5)

$$\det(F) = \det(1 + PV/m)\ . \tag{6}$$

The operator PV is presented by the finite dimensional, $2n \times 2n = 4 \times 4$, matrix. Thus the complicated fermionic functional determinant is reduced for the case considered to the very simple one.

The wave functions of the zero modes are found in terms of the AHDM construction of Ref. [6]. With their help the determinant in Eq.(5) can be evaluated explicitly. The result following from Eqs.(5),(6) has a very simple form for large separation between instantons

$$S_F \approx \frac{f^2\phi^2}{2m^2}\frac{\rho_1^2\rho_2^2}{r^4}\sin^2\gamma \ . \tag{7}$$

Here ρ_1, ρ_2 are the radii of the two instantons, r is their separation, $r > \rho_1, \rho_2$, and γ is the angle beween the directions of the orientation of the instantons defined by the identity $\mathrm{Re}(q_1^+ q_2) = \rho_1\rho_2\cos\gamma$, where q_1, q_2 are the quaternions describing the orientations and radii of instantons [6].

Note that if we consider the case when $\rho_1, \rho_2 \ll f\phi$ then the scalar condensate does not strongly disturb the instantons themselves. At the same time, according to Eq.(7) there appears the strong interaction between the instantons making their identical orientation preferable. There is no such interaction between the antiinstantons. One can reverse the situation changing the sign in front of γ_5 in Eq.(4). Then the antiinstantons interact and the instantons do not. Evaluating this result we consider the correction to the gauge field action given by the fermions in the one-loop approximation. Note that we take into account only those fermionic loops which are disturbed by the scalar condensate as well as by the gauge field. We neglect the loops of scalars and gauge fields. Nevertheless we can rely upon the obtained result. Let us keep in mind that the instanton interaction depends on the parameter $\zeta = f\phi/(2m)$ which has the fermion mass in the denominator. It makes the found correction to the action to be important for small m. Certainly ζ should not be considered as a large parameter because the scalar condensate might give a contribution to m ($\delta m \sim f\phi$), but one can consider both the scalar condensate and the fermions mass to be small, $\phi,\ m \to 0$, keeping their ratio constant: $\zeta = const$. In this limit all one-loop corrections to the action omitted in the present calculations are reduced to the ones in a pure gauge theory with the zero value of scalar condensate. They are recognised to give no interaction between the instantons, their role in the problem of interest is the renormalization

of the coupling constants and masses. Therefore in the considered limit the instanton-instanton interaction (7) proves to be valid.

The interaction (7) between the instantons allows one to address in the future the problem of the phase transition into the state in which the instantons are polarized and the antiinstantons belonging to the same $SU(2)$ gauge group are not (or inversely, the antiinstantons are polarised and instantons are not). To apply the results of this paper to the problem of gravity, as it is formulated in [1], one has to resolve the following problem. The construction considered in [1] required that the gauge field remained massless. This is necessary, in particular, to obtain massless gravitational waves in the theory. This requirement is violated in the considered model: there appears the mass of the gauge field $M_V = (3/4)g^2\phi^2$. One, heuristic, way to deal with this difficulty is provided by the discussed above limit $\phi,\ m \to 0,\ \zeta = const$, in which $M_V = 0$. Another possibility to avoid the appearance of the gauge field mass is based upon a modification of the considered model discussed elsewhere.

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